Fair Matching under Constraints
(with application to daycare match)

Street protest in Japan

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Overview

• Many matching markets are subject to constraints
  • Affirmative action (diversity constraints)
  • Gender composition in workplace
  • More real-life examples: Later
• Question: Desirable outcomes (mechanisms)?
Main Results

• Stable matching does not always exist

• **Fair matchings** are characterized via fixed points of a function

• Necessary and sufficient condition for existence of a **student-optimal fair matching (SOFM)**
  
  • **general upper-bound**

• Application to daycare/nursery school match (in progress)
Model

- **Students** (denoted \(i, I\)) and **schools** (denoted \(s, S\))
  - Many-to-one matching
- Each Student has strict **preferences** over schools (& outside option, \(\emptyset\))
- Each school has a strict **priority order** over students
  - Generalizable to weak priority (i.e., ties)
Constraints

• Each school $s$ is subject to a constraint

• For each subset $I'$ of students, a constraint tells “feasible” or “infeasible”

• c.f. Constraints at the level of sets of schools (Biro et al. 2010, Kamada and Kojima, 2015, 2016a,b, Kojima et al. 2016, Goto et al. 2016)

• For each school, assume there is at least one feasible set of students.
Desirable properties

• **Feasibility:** assignment at every school is feasible

• **IR:** no student is matched to a school ranked below ∅

• **Non-wastefulness:** there are no $i, s$, such that
  • $i$ prefers $s$ to her own assignment,
  • moving $i$ to $s$ results in a feasible matching

• **Fairness** (elimination of justified envy): there are no $i, i', s$, such that
  • $i$ prefers $s$ to her own assignment,
  • $i'$ is matched to $s$ and $i$ has a higher priority than $i'$ at $s$
Discussion on fairness

• **Fairness** (elimination of justified envy): there are no $i, i', s$, such that

  • $i$ prefers $s$ to her own assignment,

  • $i'$ is matched to $s$ and $i$ has a higher priority than $i'$ at $s$

  • and replacing $i'$ with $i$ is feasible at $s$

• Appropriate fairness concept depends on applications

  • Labor markets (medical match): weak fairness

  • College admission with disability, disaster relief material: fairness

• Non-existence problem robust to fairness concepts employed
Preliminary Facts

- Fact 1: Feasibility & IR & Fairness & Non-wastefulness $\Leftrightarrow$ stability

- Fact 2: Stability (=Feasibility & IR & Fairness & Non-wastefulness) leads to non-existence
  
  - “Necessary and sufficient” condition turns out to be capacity constraints (later)
Fair matching

• Approach: Don’t insist on (exact) non-wastefulness but require *fairness* (+ feasibility, IR)

• Existence? Structure?
  
  • Characterization via a mapping
Cutoff adjustment function

- Let \( P_s \) be the cutoff (=lowest priority/“score” to be admitted) at school \( s \);
  - regard as an element in \( \{1, \ldots, n, n+1\} \), where \( n:={\text{number of students}} \).
- Consider a cutoff profile \( P=(P_s)_s \) at all schools.
- Given \( P \), let \( D(P)=(D_s(P))_s \) be the demand profile for different schools (each student chooses favorite available school given cutoff or \( \emptyset \))
- Define **cutoff adjustment function** \( T \) from cutoff profiles to themselves by:
  - \( T_s(P)=P_s+1 \) (modulo \( n+1 \)) if \( D_s(P) \) is infeasible (i.e., “over-demanded”)
  - \( T_s(P)=P_s \) otherwise.
- \( T \) is like Walrasian tatonnement but doesn’t try to eliminate under-demand
Characterization

**Theorem:** If a cutoff profile $P$ is a fixed point of $T$, then the induced matching is feasible, individually rational, and fair. Moreover, if a matching is feasible, individually rational, and fair, then there exists a cutoff profile that induces it.

- Proof: Given $P$ induces matching $D(P) = (D_s(P))_s$,
  - there is no guarantee that $D(P)$ is feasible, but
  - $D(P)$ is IR and fair
  - $P = T(P)$ iff $D(P)$ is feasible by definition of $T$. 

Problem with fairness

- An arbitrary fair matching may be undesirable.
- Is there a “(most) desirable” fair matching?
SOFM

• A matching is a **student-optimal fair matching (SOFM)** if

  • fair, IR, feasible, and

  • weakly preferred by every student to any matching satisfying (1).

• Similar to “student-optimal stable matching” in standard case

  • note a stable matching may not exist
General upper bound

- We say constraints are **general upper-bound** if every subset of a feasible subset is also feasible
  - subsume standard settings like (1) capacity constraints and (2) type-specific quotas (diversity in schools), but exclude e.g., minimum (floor) constraints
- More (less standard) examples of general upper-bound
  - College admission with students with disability (budget constraint)
  - Refugee match (Delacretaz et al. 2016)
  - School Choice and bullying (Kasuya 2016)
  - Separating conflicting groups in refugee match
  - Daycare/nursery school matching: Later
Sufficiency for SOFM

**Theorem:** If each school’s constraint is a general upper bound, then there exists an SOFM.

- Similar to the existence of SOSM in standard case
  - note a stable matching may not exist
- Computation is easy (c.f. proof)
Proof (1)

• Given our characterization theorem, we study fixed points of T.

• Under general upper bound, use Tarski’s fixed point theorem (below)

• A set is called a **lattice** if for any pair of elements, their “join” (least upper bound) and “meet” (greatest lower bound) both exist.

  • Example: \(\{1, \ldots n+1\}^m\) with product order

  • In particular, there is a “largest” and “smallest” elements

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**Tarski’s Theorem (special case):** Let X be a finite lattice and f: X→X be weakly increasing, i.e., \(x \leq x'\) implies \(f(x) \leq f(x')\).

Then the set of the fixed points of f is a finite lattice. In particular, there are largest and smallest fixed points.
Proof (2)

Tarski’s Theorem (from last slide): Let $X$ be a finite lattice and $f: X \to X$ be weakly increasing, i.e., $x \leq x'$ implies $f(x) \leq f(x')$.

Then the set of the fixed points of $f$ is a finite lattice. In particular, there are largest and smallest fixed points.

• Back to proof: We’ll show $T$ is weakly increasing. Suppose $P \leq P'$.

  1. If $P_s < P'_s$, then $T_s(P) \leq P_s + 1 \leq P'_s \leq T_s(P')$.

  2. Suppose $P_s = P'_s$.

    • Demand for $s$ is (weakly) larger if students face higher cutoffs at all other schools, so $D_s(P)$ is a subset of $D_s(P')$.

    • So, $T_s(P) = P_s + 1$ implies $T_s(P') = P'_s + 1$, thus $T_s(P) = T_s(P')$.

    • So $T(P) \leq T(P')$.

• Smallest fixed point induces SOFM. \(\text{QED}\)
Algorithm

- Tarski’s theorem gives an intuitive (and polynomial-time) algorithm.

- Start with lowest possible cutoff profile, $P$ (i.e., every student is above the cutoff at every school)
  
  - Then $P \leq T(P)$
  
  - Apply $T$ repeatedly and get: $P \leq T(P) \leq T(T(P)) \leq T^3(P) \leq T^4(P) \leq \ldots$
  
  - At some point it stops at some $P^*$, and
    
    - $T(P^*) = P^*$; so it induces a feasible, IR, and fair matching
    
    - For any fixed point $P$, $P^* \leq P$; $P^*$ corresponds to SOFM
More general constraints?

• The “general upper-bound” includes many practical cases, but not all (e.g., minimum constraints)

• Does SOFM exist more generally?

• Answer: a qualified “no”:

**Theorem**: Suppose the constraint of a school $s$ is not a general upper bound. Then there exist student preferences and capacity constraints at other schools s.t., SOFM does not exist.
Proof (1)

• Suppose the constraint at $s$ is not a general upper bound.

• Consider two cases:
Proof (2)

• Case 1 ("easy" case): Suppose the empty matching (i.e., no one is matched) is infeasible at s.

• Assume all students find s unacceptable.

• Clearly, there is no feasible and IR matching.
Proof (3)

• Case 2 ("less easy" case): Suppose the empty matching is feasible at $s$.

• Note there is some set $I'$ of students and its subset $I''$ such that $I'$ is feasible but $I''$ is not (and both are nonempty).

• Fix $s' \neq s$ and assume preferences
  • students in $I''$: $s$, $s'$
  • students in $I' \setminus I''$: $s'$, $s$
  • all other students find all schools unacceptable
  • $s'$ has a large capacity
Proof (4)

- Two fair (&feasible and IR) matchings:
  1. everyone in $I'$ is matched to $s$ and everyone else is unmatched
  2. everyone in $I'$ is matched to $s'$ and everyone else is unmatched.

- If there is SOFM, then it should
  - match everyone in $I''$ to $s$, $I'|I''$ to $s'$
  - all other students find all schools unacceptable
  - $s'$ has a large capacity

Recall (from last slide)
- students in $I''$: $s$, $s'$
- students in $I'|I''$: $s'$, $s$
- all other students find all schools unacceptable
- $s'$ has a large capacity
Application: Daycare Match

- Some resources (teachers, rooms, etc.) can be used for kids of different ages (Okumura 2017)
- Resource demand per kid varies across ages (younger kids need more teachers and space per capita)
  → general upper bound (but not capacity)

- Japan: Daycare (nursery school) is greatly over-demanded
- Municipal governments are under great pressure to accommodate more children
  - flexible assignments tried in several municipalities (but in ad hoc manners)
- Centralized matching algorithms.
Comparative statics

**Proposition:** SOFM under flexible constraints is Pareto superior for students to SOFM under rigid constraints.

- Easy to prove, true more generally for arbitrary “relaxation of constraints”

- c.f. Results for SOSM in standard models (e.g., Crawford 1991; Konishi and Unver 2006)

- Flexibility across different ages will help.
  - How about the magnitude?
Daycare Match Simulation (in progress)

• Data from city of Yamagata (Yamagata) and Bunkyo Ward (Tokyo), Japan:
  • preferences (mechanism is strategy-proof)
  • priorities
  • outcomes
• We simulate SOFM under “flexible” and “rigid” constraints

Recall: SOFM under flexible constraints is Pareto superior to SOFM under rigid constraints.
Matched Children

(Data: Yamagata)
Rank distribution

**Rank Distribution**

- SOFM rigid
- SOFM flexible
- Actual outcome

**SOSM flexible v. actual**

(Data: Yamagata)
Extension: Tie in priority

• College admission in Hungary (Biro 2010) uses a mechanism like deferred acceptance, but
  • Ranking over students are based on test score → ties
  • Admitting all students with a score is infeasible → reject all students of that score

• Disaster shelter in Kobe and Tohoku earthquakes (Hayashi 2003, Hayashi 2011)
  • Priorities include lots of ties (e.g., own house livable or not)
  • Insufficient food supply was not allocated
Problems with ties:

- A has capacity of 1
- A ranks 1 and 2 equally
- But our theory extends: SOFM exists, etc.

- Characterization: fair and non-wastefulness are compatible iff capacity constraints and no ties.
Stability: Maximal domain

- Recall stability (=Feasibility & IR & Fairness & Non-wastefulness) leads to **non-existence**. In fact,

**Theorem:** Suppose the constraint of a school $s$ is not a capacity constraint (while being a general upper-bound). Then there exist a priority at $s$ and student preferences s.t. there exists no stable matching.

- Note: “necessary and sufficient” condition for stable matching existence
- The conclusion holds for any priorities and constraints at other schools.
Strategic issues

• SOFM mechanism isn’t necessarily strategy-proof for students

• Capacity constraints $\rightarrow$ SP for students

• Turns out this is “necessary” as well.

**Theorem:** Suppose the constraint of a school $s$ is not a capacity constraint. Then there are school priorities and standard capacity constraints at other schools such that the SOFM mechanism isn’t strategy-proof for students.

• But

  • The same impossibility holds for any mechanism with feasibility, fairness, and *unanimity*.

  • Approximate incentive compatibility holds in large markets.
Related literature


• **Fairness:** Foley (1967), Balinski-Sonmez (1999 JET), Sotomayor (1996 GEB), Blum-Roth-Rothblum (JET 1997), Wu-Roth (2017 GEB), Kesten-Yacizi (2010 ET), Biro (2010)
Conclusion

• Characterization of fair matchings via a cutoff adjustment function

• The general upper-bound is the most general condition to guarantee existence of SOFM

• Daycare match application

• Future research
  • Solution under non-general upper bounds
  • More numerical and empirical study
  • Implementing the design